A MATHEMATICAL PROCEDURE ON THE SELF-ORGANIZATION PROCESS IN BEES

Olatinwo, Memudu Olaposi

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.
Email: molaposi@yahoo.com

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Abstract

A mathematical procedure on the self-organization process involving the colony of bees which results in the construction of the honey-comb having hexagonal cross section and optimal capacity is presented in this paper. The optimal capacity of the honey comb as well as the bee farmer's profit are also considered. We employ the notion of the distance function in our presentation.

Keywords: Self-organization process, distance function, honey-comb, hexagonal cross section, colony of bees, optimal capacity.

Introduction

In nature, there exists the phenomenon of systems which at a stage look certain to be going to end up in a state of confusion, chaos, uncertainty or instability but just emerges into a state of order, stability, regularity without a visible control agent or external influence being responsible for this turn around. These are the systems being referred to by some cyberneticians as 'self-organizing systems'

A self-organizing system is one which controls itself implicitly. One of the meanings of self-organizing systems is that a "self-organizing system is one that changes itself from a bad way of behaving to a good one". This meaning is particularly interesting to us because of the word "itself" in the definition. There is no separate control unit that effects the change. The system changes itself. For this (Ashby's) definition of self-organizing system, we refer to Adeagbo-Sheikh, (2003).

According to Beer, (1978), in his definition of self-organizing systems, "in nature, the structure of control -its effective organization- is not monitored by a pantheon of directors which decide to change the structure. The structure just changes". Adeagbo-Sheikh, (2003) gave a very general conceptual model on the theory of self-organizing systems. In his model, "a self-organizing system is a dynamical system such that whenever its trajectory includes a certain point A, then it must (even in the face of disturbances) later include a certain point B". Beer, (1978) cited concrete examples of self-organizing systems. For instance, the case of certain species like the cabbage aphides which, by the rate at which they reproduce could overrun the earth but are maintained in nature at a well manageable population level. Also, the human form which in every case develops with parts growing in definite stable proportions. The bees by their jostling, falling and random movements construct honeycombs that are hexagonal in cross section and of optimal design as regards capacity as if the bees were engineers that calculate and employ sophisticated instruments that produce this regular design. There are many other examples of self-organizing systems cited in Beer, (1978). However, in this paper, we shall be interested in presenting a mathematical procedure on the self-organization process which results in the construction of an honeycomb of hexagonal cross section and optimal capacity. The optimal capacity of the honeycomb as well as the farmer's profit are also considered in our presentation.

Actually, the bees are prudent planners as they always plan ahead of (tough) time. Due to the fact that their movement is restrained during the rainy season, the bees prepare and store food safely during the dry season for the rainy season. This food is known as honey. It is interesting to remark that the bee colony is well-organized in the sense that the bees employ the principle of division of labour in carrying out their activities during honey production. In the colony, there is the queen, the workers and the duly committed soldiers. Whenever there is an external attack, the soldiers are battle-ready to wage war so as to resist the attack. While some workers look for flower nectar and other ingredients necessary for honey production, others fetch water for the same reason. Moreover, man has remained as the co-competitor with the bees in the sharing of honey produced because of its medicinal and economic values to the mankind. Various methods of beekeeping for honey production are available. For instance, see Killion, (1951), Rope, (1962) and Savage, (1961).

The working functions for the self-organization process involving the colony of bees which results in the construction of an honeycomb of optimal capacity and hexagonal cross section will be determined in the next section.

Determination of the working functions for the self-organization process involving the bee colony.

We first recall the following definitions:

Definition 2 .1 : The distance function $(g(t))$ is the distance from the goal at time(t) satisfying the following properties:

(i) $g(t_o) > 0 \ , t_0 \geq 0$

(ii) $g'(t) < 0 \ , 0 \leq t_0 < t < t_1 < \infty$

(iii) $g(t_1) = 0, 0 \leq t_0 < t_1 < \infty$

(iv) $|g'(t)| < \infty$

See Adeagbo-Sheikh, (2003) for this definition.

Definition 2.2 : The working functions are those functions whose values correspond to the attainment of self-organization during a self-organization process. See also Adeagbo-Sheikh, (2003).

We now formulate the problem as follows:

Since the cross section of an honeycomb is characterized by a number of hexagons, then the volume (V) of an honeycomb becomes a function of its height (h) and the area of the entire number of hexagons. We assume that the hexagons are regular throughout this paper. Let the length of each side of the hexagon be x.

Volume (V) = area of base $\times$ perpendicular height
i.e V = Ah (2.1)

where A is the area of the entire number of the hexagons and A is taken as the base area of the honeycomb; h is the thickness of the honeycomb and it is regarded as the perpendicular height.

area of an hexagon = 6. ½ x^2sin 60^o

$$= \frac{3\sqrt{3}}{2}x^2$$

Therefore, A = $\frac{3\sqrt{3}}{2}nx^2$, where n is the entire number of hexagons in an honeycomb.

We assume that the thickness of each hexagon is negligible. Thus, the volume of an honeycomb is given by:

$$V = \frac{3\sqrt{3}}{2}nx^2 h, \; x \geq 0, h \geq 0, \; n \geq 0,$$

$$i.e \; V = k x^2 h, x \geq 0, h \geq 0 \qquad (2.2)$$

$$where \; k = \frac{3\sqrt{3}}{2}n, k \geq 0,$$

The total surface area (S) of the honeycomb is assumed to be a linear combination of *x* and *h*. That is,

$$S = \alpha x + \beta h, \alpha > 0, \beta > 0, \qquad (2.3)$$

where α and β are constants having the dimensions of the length. Moreover, α and β can be determined experimentally. A suitable experiment that can be suggested is to measure *S*, *h* and *x* for various honey-combs from which a suitable graph can be plotted. The values of α and β are then determined as the gradient and intercept of the line of best fit on the vertical axis.
From eqn (2.3),

$$h = \frac{S - \alpha x}{\beta} \qquad (2.3a)$$

By parametrizing eqn (2.3a) in t, we obtain

$$h(t) = \frac{S(t) - \alpha x(t)}{\beta} \qquad (2.3b)$$

where t is the time variable, $h(t)$ does not grow indefinitely with time. The function $h(t)$ in eqn (2.3b) is called the distance-from-goal expression for the colony of bees which is self-organizing to construct an honeycomb of hexagonal cross section and optimal capacity (see Adeagbo-Sheikh 2003). Our assignment is to choose the functions $S(t)$ and $x(t)$ so that eqn (2.3b) becomes a distance function. This will be the distance function corresponding to the colony of bees which is self-organizing to construct an honeycomb having hexagonal cross-section and of optimal capacity.

$S(t)=S^*(t)$ and
$x(t) = x^*(t)$ so obtained under this situation will be called the working functions for the self-organization process.

In order to be able to find the working functions for the self-organization process, we must recall the following results.

Theorem 2.1 : Let $g(t)$ be a smooth function of t. Then, $g(t)$ is a distance function for a self-organizing system, denoted by $stt(t_o,t_1)$, if and only if $g(t)$ is expressible in the form

$$g(t) = (t_1-t)\omega(t) \qquad (2.4)$$

where $\omega(t)$ is a smooth, positive function in the open interval (t_0,t_1) such that $\omega(t)>(t_1 - t)\omega^1(t)$.

Corollary 2.1: A sufficient condition for smooth function $g(t)$ to be a distance function for a self-organization process $stt(t_0,t_1)$ is that $g(t)$ be expressible in the form

$$g(t) = (t_1-t)\omega(t),$$

where $\omega(t)$ is a smooth, positive monotone decreasing function in (t_0,t_1).

See Adeagbo-Sheikh (2003) for the proofs of the Theorem and its corollary.

Re-writing eqn (2.3b) in the from of eqn (2.4) yields

$$h(t) = (t_1 - t)\left(\frac{S(t) - \alpha x(t)}{\beta(t_1 - t)}\right) \qquad (2.5)$$

The 2-jet of $\dfrac{1}{t_1 - t}$ with the constant term is given by

$$\frac{1}{t_1 - t} = \frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3} \qquad (2.6)$$

See Bruce and Giblin (1992) for detail on the k-jet of functions.

Type (2.6) equation is often made use of in equation type (2.5) to avoid singularity at $t=t_1$. Thus, the transformed function becomes

$$H(t)=\frac{1}{\beta}(t_1-t)\left\{\left(\frac{1}{t_1}+\frac{t}{t_1^2}+\frac{t^2}{t_1^3}\right)(S(t)-\alpha x(t))\right\} \tag{2.7}$$

where $\omega(t)=\frac{1}{\beta}\left(\frac{1}{t_1}+\frac{t}{t_1^2}+\frac{t^2}{t_1^3}\right)(S(t)-\alpha x(t))$, $0 \le t_0 < t < t_1$.

The first main result, herein given as Theorem 2.2, provides the conditions under which eqn (2.7) represents the distance function for the colony of bees which is self-organizing to construct honeycomb of hexagonal cross section and of optimal capacity.

Theorem 2.2: The function $H(t)$ defined by eqn(2.7) becomes a distance function for the colony of bees which is self-organizing to construct honeycomb of hexagonal cross section and optimal capacity if the following conditions hold:

- *(i)* $H(t_o)>0,\ t_0 \ge 0$
- *(ii)* *If* $H(t)$ is a positive monotone decreasing function, then

  $$(2t+t_1)\,|S(t)-\alpha x(t)| < (t^2+t_1 t+t_1^2)\,|S'(t)-\alpha x'(t)|,$$

  $0 \le t_0 < t < t_1 < \infty$;
- *(iii)* $H(t_1)=0,\ 0 \le t_0, < t_1 < \infty$;
- *(iv)* $S(t)$ is a positive monotone decreasing function *;*
- *(v)* $x(t)$ is a positive monotone increasing function ;
- *(vi)* *$x(t)$ is bounded above by* $|S(t)|/\alpha$, *i.e* $|x(t)| < \frac{|S(t)|}{\alpha}$.

Proof (i): From eqn (2.7),

$$H(t_0)=\frac{1}{\beta}(t_1-t_o)\left(\frac{1}{t_1}+\frac{t_o}{t_1^2}+\frac{t_o^2}{t_1^3}\right)(S(t_o)-\alpha x(t_o)),\ t_o \ge 0$$

Since $\beta>0$, $t_0 < t_1$, $\frac{1}{t_1}+\frac{t_o}{t_1^2}+\frac{t_o^2}{t_1^3}>0$, then $H(t_0)>0$ when $S(t_0)-\alpha x(t_0)>0$,

that is,

$H(t_0)>0$ when $x(t_0) < \frac{S(t_o)}{\alpha}$, $\alpha>0$, thus proving (i).

Proof (ii): Observe that

$$\omega'(t) = \frac{1}{\beta}\left\{\left(\frac{1}{t_1^2} + \frac{2t}{t_1^3}\right)\left(S(t) - \alpha x(t)\right) + \left(\frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3}\right)\left(S'(t) - \alpha x'(t)\right)\right\}$$

Since $t_1 - t > 0$ in eqn (2.7) for $t < t_1$, then H(t) is a positive monotone decreasing function when ω(t) is a positive monotone decreasing function by corollary (2.1). Therefore, differentiating eqn(2.7) yields

$$H'(t) = \frac{1}{\beta}[-\left(\frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3}\right)(S(t) - \alpha x(t)) + (t_1 - t)\left(\frac{1}{t_1^2} + \frac{2t}{t_1^3}\right)(S(t) - \alpha x(t)$$

$$+ (t_1 - t)\left(\frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3}\right)(S'(t) - \alpha x'(t))]$$

$$=-\frac{1}{\beta}\left(\frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3}\right)(S(t) - \alpha x(t)) + (t_1 - t)\times$$

$$\left[\frac{1}{\beta}\left\{\left(\frac{1}{t_1^2} + \frac{2t}{t_1^3}\right)(S(t) - \alpha x(t)) + \left(\frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3}\right)(S'(t) - \alpha x'(t))\right\}\right]$$

$= -\omega(t) + (t_1 - t)\,\omega'(t)$

< 0, since $\omega(t) > 0$ *for* $S(t) - \alpha x(t) > 0$,

and $\omega'(t) < 0$.

for $\left(\frac{1}{t_1^2} + \frac{2t}{t_1^3}\right)(S(t) - \alpha x(t)) + \left(\frac{1}{t_1} + \frac{t}{t_1^2} + \frac{t^2}{t_1^3}\right)(S'(t) - \alpha x'(t)) < 0$

i.e $(t_1 + 2t)(S(t) - \alpha x(t)) + (t^2 + t_1 t + t_1^2)(S'(t) - \alpha x'(t)) < 0$.

$(t_1 + 2t)(S(t) - \alpha x(t)) > 0$, $(t^2 + t_1 t + t_1^2)(S'(t) - \alpha x'(t)) < 0$,

since $t_1 + 2t > 0$, $(S(t) - \alpha x(t)) > 0$, $(t^2 + t_1 t + t_1^2) > 0$.

It implies that $(S'(t)-\alpha x'(t))$. must be negative i.e $(S'(t)-\alpha x'(t)) <0$.
Therefore,

$$(t_1+2t)(S(t)-\alpha x(t)) < -(t^2 + t_1 t + t_1^2)(S'(t)-\alpha x'(t)) .$$

so that

$$|(t_1+2t)(S(t)-\alpha x(t))| < |-(t^2 + t_1 t + t_1^2)(S'(t)-\alpha x'(t))|$$

$$|(t_1+2t)||(S(t)-\alpha x(t))| < |-(t^2 + t_1 t + t_1^2)||(S'(t)-\alpha x'(t))|$$

i.e $|(t_1+2t)||(S(t)-\alpha x(t))| < (t^2 + t_1 t + t_1^2)| \; (S'(t)-\alpha x'(t))|$

Hence, if $H(t)$ is a positive monotone decreasing function then

$$(2t+t_1)|S(t)-\alpha x(t)| < (t^2 + t_1 t + t_1^2)\,|(S'(t)-\alpha x'(t))|$$

This concludes the proof of (ii).

Proof (iii) : putting $t = t_1$ in eqn (2.7) yields $H(t_1)=0$, thus proving condition (iii).

Proof (iv) & (v) : From the proof of (ii) above ,

$\omega(t)>0$ when $S(t)-\alpha x(t)>0$ and $\omega'(t) <0$ when $S'(t)-\alpha x'(t)<0$.

We assume that $S(t)>0$ *and* $x(t)>0$ *for* $S(t)-\alpha x(t)>0$ gives $H(t)>0$, *or* $\omega(t) >0$.

Also, $S'(t)<0$, $x'(t)>0$, $\alpha>0$, *since* $S'(t)-\alpha x'(t)<0$. It follows that $S(t)$ is positive monotone decreasing while $x(t)$ is positive monotone increasing.
Proof (vi): Since $S(t)-\alpha x(t)>0$, then $x(t) < \dfrac{S(t)}{\alpha}$ from which we obtain $|x(t)| < \dfrac{|S(t)|}{\alpha}$

$t_0 < t < t_1 < \infty$.

Hence, the proof of the theorem is complete.

The next result, which follows from Theorem 2.2, guarantees the choice of the working functions for the colony of bees which is self-organizing to construct honeycomb of hexagonal cross section and optimal capacity.

Theorem 2.3: Suppose that in the expression for $H(t)$ in eqn(2.7), the following choices are made:

$S(t)=S^*(t)=c_1 t^{\gamma}$, $c_1>0$, $\gamma<0$ $x(t)=x^*(t) = c_2 t^m$, $m>0$, $c_2>0$. Then, *H(t)* in eqn (2.7) reduces to a distance function for the self-organization process of the colony of bees resulting in the construction of an honeycomb of hexagonal cross section and of optimal capacity.

Proof: Substituting for *S(t)* and *x(t)* in eqn (2.7) yields

$$l(t)=(t_1-t).\ \frac{1}{\beta}\left\{\left(\frac{1}{t_1}+\frac{t}{t_1^2}+\frac{t^2}{t_1^3}\right)\left(c_1 t^{\gamma}-\alpha c_2 t^m\right)\right\},\ 0<t_o<\ t<t_1 \qquad (2.8)$$

where $\omega(t)\ =\ \frac{1}{\beta}\left(\frac{1}{t_1}+\frac{t}{t_1^2}+\frac{t^2}{t_1^3}\right)\left(c_1 t^{\gamma}-\alpha c_2 t^m\right)$, $\omega(t)\ >0,$

$\omega'(t)\ <0$ for

$$\left(\frac{1}{t_1^2}+\frac{2t}{t_1^3}\right)\left(c_1 t^{\gamma}-\alpha c_2 t^m\right) < -\left(\frac{1}{t_1}+\frac{t}{t_1^2}+\frac{t^2}{t_1^3}\right)(\gamma c_1 t^{\gamma-1}-\ m\alpha c_2 t^{m-1}),$$

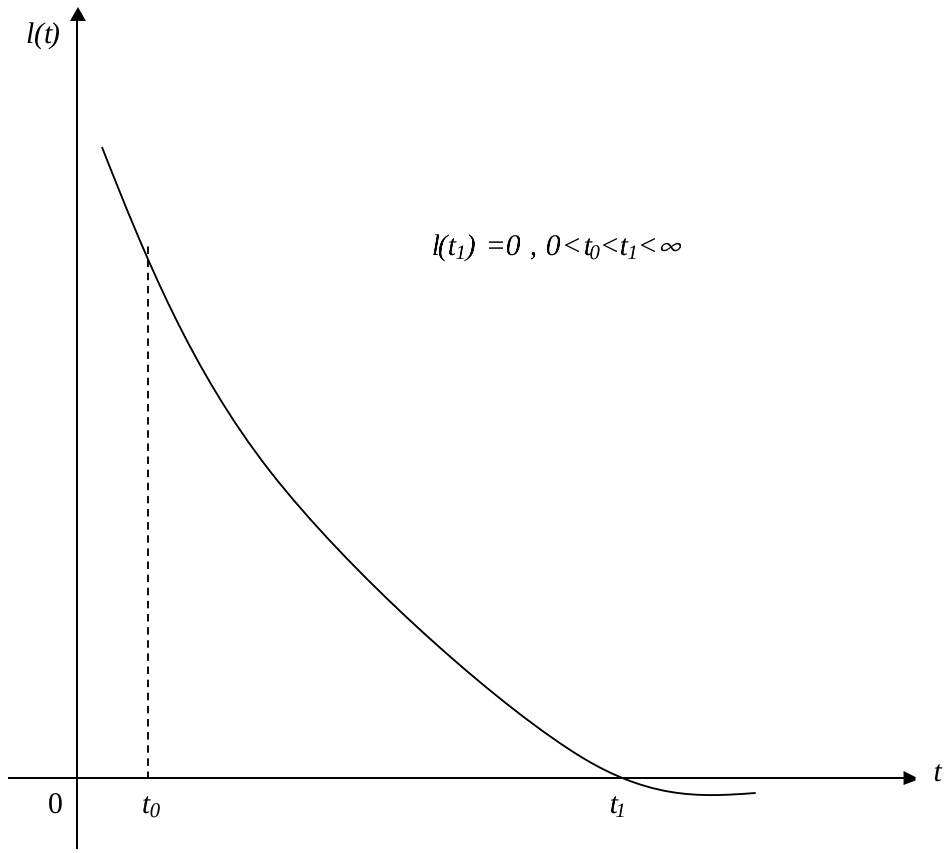
$\omega(t)\ >0$ for $c_1>\alpha\ c_2 t^{m-\gamma}$.

Therefore, by corollary (2.1), *l(t)* is in the form

$g(t)=\ (t_1-t)\omega(t).$

So, *l(t)* is a distance function for the colony of bees which is self- organizing to construct an honeycomb of hexagonal cross section and optimal capacity. Hence, the choice $S^*(t)=c_1t^{\gamma},\ c_1>0,\ \gamma<0;\ x^*(t)=c_2t^m,\ c_2>0,\ m>0$ appropriately represents the working functions for the self-organization process.

The sketch of the graph of *l(t)* is shown in the figure 2.1 below.



The Graph of Distance function *(l(t))* against time(t)

Figure 2.1

The next section is devoted to the determination of the optimal capacity of the honeycomb obtained under this problem as well as investigating on the profitability of the honey business to the bee-farmers (beekeepers).

Determination of the Honeycomb Capacity and the Bee-Farmer's profit

We shall begin by finding the maximum and minimum values of the capacity of the honeycomb whose capacity and surface area are defined in eqns(2.2) and (2.3) respectively. Substituting eqn (2.3a) in eqn (2.2) yields

$$V = \frac{k}{\beta}(Sx^2 - \alpha x^3) \qquad (3.1)$$

$k \geq 0$, $\alpha > 0$, $\beta > 0$ and S is assumed to be fixed.

Differentiating V with respect to x in eqn (3.1) and setting $\frac{dV}{dx} = 0$ for the critical points yield.

$$x = 0, \qquad \text{or}, \qquad x = \frac{2S}{3\alpha}$$

We have $\frac{d^2V}{dx^2} = \frac{k}{\beta}(2S - 6\alpha x)$

$\frac{d^2V}{dx^2} < 0$ when $x = \frac{2S}{3\alpha}$ giving the maximum point (x_{max})

as $x_{max} = \frac{2S}{3\alpha}$

and the maximum volume (V_{max}) is given by

$$V_{max} = \frac{4kS^3}{27\alpha^2\beta}.$$

Similarly,

$\frac{d^2V}{dx^2} > 0$ when $x = 0$ giving the minimum point (x_{min}) as $x_{min} = 0$ and the minimum volume is given by $V_{min} = 0$.

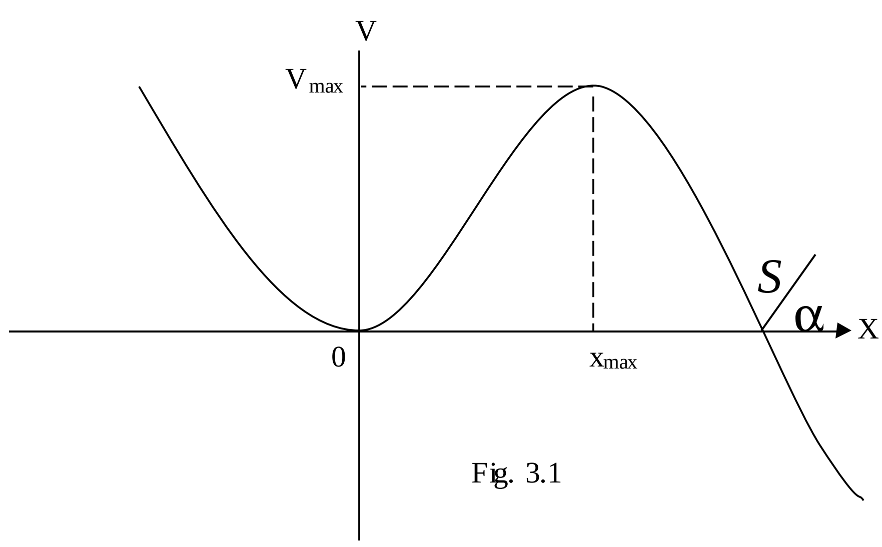


Fig. 3.1

Again,

$$V_{max} = \frac{4kS^3}{27\alpha^2\beta} \qquad (3.2)$$

Eqn (3.2) suggests that the maximum volume can be increased if k (i.e. n) is increased, α and β decreased, as well as having larger surface area. These factors suggest that a bee-farmer has to get several hives in order to increase his output. However, an approximate formula for the total yield obtained after the nth harvest is determined here.

Let $\{u_k\}_{k=0}^{n}$ be a sequence of successive yields obtained by a bee-farmer. Define this sequence by

$$u_{k+1} = au_k\ ,\ u_0 > 0,\ a > 1 \qquad (3.3)$$

Eqn(3.3) is valid since the business of honey production is very lucrative, that is,

$u_{k+1} > u_k$. Eqn (3.3) is a first order difference equation (see Wylie, 1966). On solving eqn (3.3) we obtain

$$u_n = a^n u_0,\quad n \geq 0 \qquad (3.4)$$

Eqn (3.4) is a geometric sequence having the sum

$$\begin{aligned} S_n &= \sum_{k=0}^{n} u_k \\ &= \sum_{k=0}^{n} a^k u_o \\ &= u_o \sum_{k=0}^{n} a^k \\ &= \left(\frac{a^{n+1} - 1}{a - 1}\right) u_o \qquad (3.5) \end{aligned}$$

Since $a > 1$, $S_n \longrightarrow \infty$ as $n \longrightarrow \infty$,(n is time) that is, the total yield increases infinitely as the time increases infinitely. In reality, the sum in eqn (3.5) should converge to some limit since the comb or hive cannot grow indefinitely and also the hive or comb cannot last for ever. Therefore, for a finite time (i.e n), we have a finite total yield (S_n). Eqn (3.5) gives the total yield of the bee-farmer after the nth harvest. If we assume that u_o is the initial capital invested on the business, then we can interpret eqn (3.5) as the total amount realized after the nth harvest, taking u_o as the initial capital. It is easy to see that $S_o = u_o$, $S_1 = u_o + au_o$ and so on.

Conclusion

A self-organization process involving the colony of bees was presented in this paper. The behaviours of the working functions were obtained. It was found that each side of the hexagons on the cross section of the honey comb should be a positive monotone increasing function.

Furthermore, the optimal capacity of the honeycomb constructed by the bees during the self-organization process was obtained and can be employed to advise the bee-farmers to have several hives for increased output. The total yield after the nth harvest was also obtained. It is clear from eqn(3.5) that huge profit awaits any farmer who ventures into the business of honey production. It is planned that dialogue with some business experts in honey production will be embarked upon to ascertain the degree to which our theory agrees with practical situations.

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